Answers to the Exam of Symmetry in Physics of February 4, 2011

Exercise 1

Consider the dihedral group D_3: gp$\{c, b\}$ with $c^3 = b^2 = (bc)^2 = e$.

(a) Show that the cyclic group C_3 forms an invariant subgroup of D_3.

$C_3 = \{e, c, c^2\}$ is a subgroup because it forms a subset of D_3 that is closed under the group multiplication. It is invariant since under conjugation $bcb^{-1} = c^2$, $(bc)c(bc)^{-1} = c^2$, $(bc^2)c(bc^2)^{-1} = c^2$, hence $\forall h \in C_3, \forall g \in D_3 : ghg^{-1} \in C_3$.

(b) Show that $D_3/C_3 \cong C_2$.

The cosets of C_3 in D_3: $eC_3 = cC_3 = c^2C_3$ and $bC_3 = bcC_3 = bc^2C_3$. These two cosets form a group of order 2, through the standard multiplication of cosets: $gC_3 * hC_3 = ghC_3, \forall g, h \in D_3$, e.g. $bC_3 * bC_3 = b^2C_3 = C_3$. The group $(\{eC_3, bC_3\}; *)$ is isomorphic to C_2 upon equating eC_3 with $e \in C_2$ and bC_3 with $c \in C_2$.

(c) Construct the character tables of C_2, C_3, and D_3.

C_2 and C_3 are Abelian, hence they have 2 and 3 classes respectively. Therefore, C_2 has two and C_3 three irreps. Using orthonormality of characters and that the character of a 1-dimensional irrep forms a homomorphism, together with the trivial irrep, one deduces:

C_2	(e)	(c)
$D^{(1)}$	1	1
$D^{(2)}$	1	-1

C_3	(e)	(c)	(c^2)
$D^{(1)}$	1	1	1
$D^{(2)}$	1	ω	ω^2
$D^{(3)}$	1	ω^2	ω

Here $\omega = \exp(2\pi i/3)$. Idem for the non-Abelian group D_3 which also has three classes and irreps (explicitly done in the lecture notes):

D_3	(e)	(c)	(b)
$D^{(1)}$	1	1	1
$D^{(2)}$	1	1	-1
$D^{(3)}$	2	-1	0

(d) Show which irreps of C_2 can be lifted to irreps of D_3 through $D^G(g) := D^{G/N}(gN)$.

From 1b we know that $N = C_3$, equating $eC_3 = \{e, c, c^2\}$ with $e \in C_2$ and $bC_3 = \{b, bc, bc^2\}$ with $c \in C_2$. In order for $D^{D_3}(g) = D^{D_3/C_3}(gC_3)$ to hold, $D^{D_3}(e) = D^{D_3}(c) = D^{D_3}(c^2) = D^{C_2}(e) = 1$ should hold and also $D^{D_3}(b) = D^{D_3}(bc) = D^{D_3}(bc^2) = D^{C_2}(c) = \pm 1$, which is true for the two 1-dimensional irreps of D_3. Hence, both irreps of C_2 can be lifted to irreps of D_3.

(e) Determine the Clebsch-Gordan series of the direct product rep $D^{(3)} \otimes D^{(3)}$ of D_3, where $D^{(3)}$ denotes the two-dimensional irrep of D_3.

Find the a_μ in $D^{(3)} \otimes D^{(3)} \sim \sum_{\mu=1}^{3} a_\mu D^{(\mu)}$.
The characters of $D^{(3)} \otimes D^{(3)}$:

D_3	(e)	(c)	(b)
$D^{(3)} \otimes D^{(3)}$	4	1	0

Direct inspection shows $D^{(3)} \otimes D^{(3)} \sim D^{(1)} \oplus D^{(2)} \oplus D^{(3)}$. Alternatively one can calculate $a_\mu = \langle \chi^{(\mu)}, \chi^{(3)\otimes(3)} \rangle$, e.g. $\langle \chi^{(3)}, \chi^{(3)\otimes(3)} \rangle = (1 \cdot 2 \cdot 4 + 2 \cdot -1 \cdot 1 + 0)/6 = 1$.

Consider a molecule with D_3 symmetry. By application of an external magnetic field the symmetry is broken to a C_3 symmetry.

(f) Decompose the irreps of D_3 into those of C_3 using the character tables.

Using the character table (direct inspection or orthonormality) and $\omega + \omega^2 = -1$ one finds:

$$\begin{aligned} D^{(1)}_{D_3} &= D^{(1)}_{C_3} \\ D^{(2)}_{D_3} &= D^{(1)}_{C_3} \\ D^{(3)}_{D_3} &= D^{(2)}_{C_3} \oplus D^{(3)}_{C_3} \end{aligned}$$

(g) Explain what the implications of the symmetry breaking are for the degeneracy of the energy levels of the molecule.

The degeneracy of the states transforming according to the 2-dimensional irrep $D^{(3)}$ of D_3 is generally lifted. After symmetry breaking the 2-dimensional invariant subspace splits into two 1-dimensional invariant subspaces, that are not related by the C_3 symmetry.

Exercise 2

(a) Explain what the concept of symmetry means in physics.

> Symmetry is invariance under a set of operations. In physics these are operations that leave the physical system, such as a Hamiltonian, invariant.

(b) Explain the role of representations in physics.

> Physical systems are described by quantities such as vectors, tensors, states/wave functions, or operators. These quantities transform in certain ways under the symmetry operations, which means that they transform according to certain representations of a symmetry group.

(c) Under which symmetry transformations is the Hamiltonian $H = \vec{p}^{\,2}/2m + V(|\vec{r}|)$ invariant?

> $\vec{p}$ and $\vec{r}$ are vectors, so under rotations they transform according to the vector representation D^V of $SO(3)$, but $\vec{p}^2$ and $|\vec{r}| = \sqrt{\vec{r}^2}$ are invariant under rotations. Also under reflections H is invariant, hence it is $O(3)$ invariant. Under translations the potential is not invariant.

(d) Show that $[H, U(g)] = 0$ implies that transformed states $U(g)\psi$ are degenerate in energy with the state ψ.

> If $H\psi = E\psi$, then $[H, U(g)]\psi = 0$ implies $H(U\psi) = E(U\psi)$.

(e) Describe under which representations of $O(3)$ the following quantities transform: 1) an electric field $\vec{E}$; 2) a magnetic field $\vec{B}$; 3) $\vec{E} \cdot \vec{B}$; and 4) $\vec{E} \times \vec{B}$.

> 1) An electric field $\vec{E}$ transforms as a vector under $O(3)$: D^V
> 2) A magnetic field $\vec{B}$ transforms as an axial vector under $O(3)$: D^A.
> 3) The inner product of a vector and an axial vector such as $\vec{E} \cdot \vec{B}$ transforms as a pseudo-scalar, which means it transforms trivially under rotations and picks up a minus sign under reflections: the irrep given by the determinant of the defining rep.
> 4) The outer product of a vector and an axial vector such as $\vec{E} \times \vec{B}$ transforms as vector (recall the Poynting *vector*): D^V

Exercise 3

Consider the special linear group $SL(2, \mathsf{R})$ of real 2×2 matrices with determinant 1 and its Lie algebra $sl(2, \mathsf{R})$.

(a) Give an explicit representation of the generators $a_i \in sl(2, \mathsf{R})$.

> $\exp(ta_i) \in SL(2, \mathsf{R})$ implies Tr $a_i = 0$, which implies that a general element in the Lie algebra is of the form (with $a, b, c \in \mathsf{R}$):
>
> $$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$
>
> Therefore an explicit representation of the generators is:
>
> $$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(b) Determine the dimension of $sl(2, \mathsf{R})$.

> From 3a one sees the dimension is three.

(c) Determine the center of $SL(2, \mathsf{R})$ (it is allowed to assume the defining rep is an irrep).

> Schur's lemma implies that the center of $SL(2, \mathsf{R})$ consists of matrices that are of the form:
>
> $$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$
>
> For these matrices to be in $SL(2, \mathsf{R})$, λ^2 must be 1. Hence, the center consists of the two matrices:
>
> $$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$$

(d) Show whether the map ϕ from the general linear group $GL(2, \mathsf{R})$ into $SL(2, \mathsf{R})$, given by $\phi(A) = A/\sqrt{\det A}$, is a homomorphism or not.

$$\phi(A)\phi(B) = \frac{A}{\sqrt{\det A}} \frac{B}{\sqrt{\det B}} = \frac{AB}{\sqrt{\det AB}} = \phi(AB)$$

(e) To which group is the factor group $GL(2, \mathsf{R})/SL(2, R)$ isomorphic?

$SL(2, \mathsf{R})$ is the kernel of $\det : GL(2, \mathsf{R}) \to \mathsf{R}\backslash\{0\}, A \mapsto \det A$. Hence, the factor group $GL(2, \mathsf{R})/SL(2, R) \cong (\mathsf{R}\backslash\{0\}; \times)$.

(f) Show whether $SO(2)$ is an invariant subgroup of $SL(2, \mathsf{R})$ or not.

If $AOA^{-1} \in SO(2), \forall O \in SO(2), \forall A \in SL(2, \mathsf{R})$, then $SO(2)$ would be an invariant subgroup. But in general $(AOA^{-1})^T(AOA^{-1}) \neq \mathbf{1}$ because $A^T A \neq \mathbf{1}$.